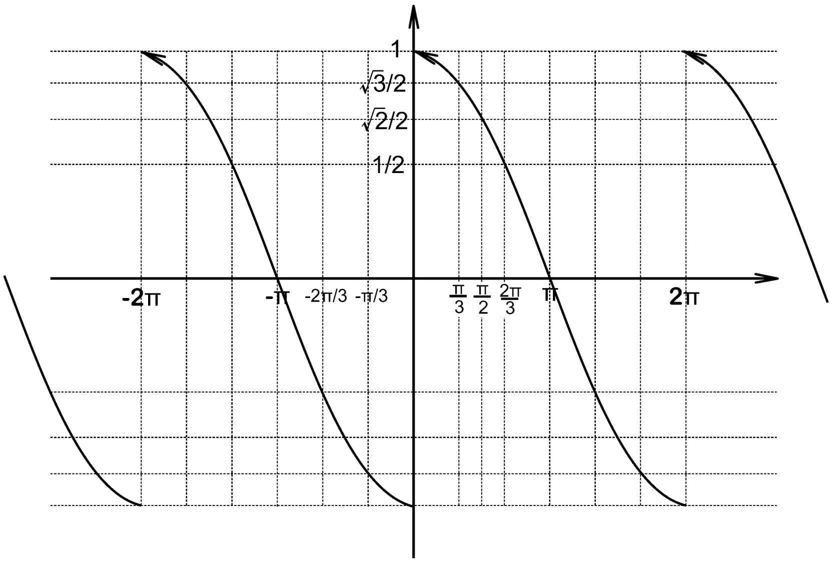
Pismeni ispit iz Analize III, 01.10.2013.
ispit pisati isključivo hemijskom olovkom



1. Funkciju definisanu grafikom razviti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje reda

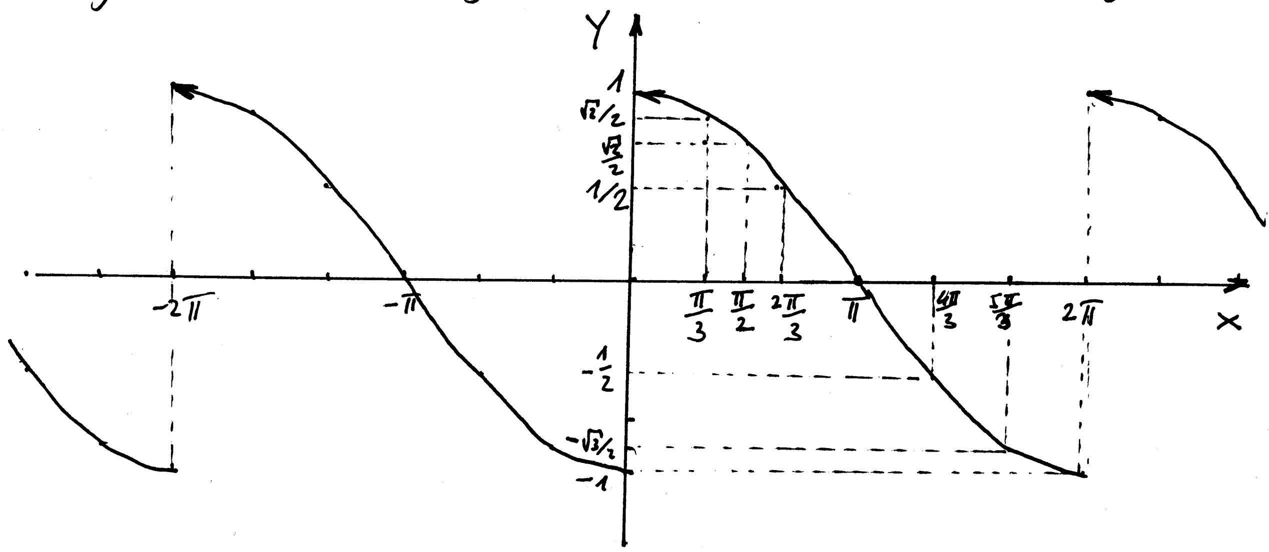
$$\frac{1}{1\cdot 3} - \frac{3}{5\cdot 7} + \ldots + \frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)} + \ldots .$$

2. Izračunati $\iint\limits_D x\,\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) : 1 \le x^2 + y^2 \le 2y,\ x \le y,\ x \ge 0\}$.

3. Izračunati krivoliniski integral druge vrste $I = \oint\limits_C (y-z)\mathrm{d}x + (z-x)\mathrm{d}y + (x-y)\mathrm{d}z$ gdje je C krug $x^2 + y^2 + z^2 = a^2$ $(a > 0)$, $y = x\,\mathrm{tg}\,\alpha$, $(0 < \alpha < \frac{\pi}{2})$ uzet u smjeru suprotnom kretanju kazaljnke na satu ako se posmatra sa pozitivnog dijela x-ose.

4. Odrediti površinu koju cilindar $x^2 + y^2 = ax$ isjeca na lopti $x^2 + y^2 + z^2 = a^2$ iznad ravni Oxy.

Funkciju definisanu grafikom razviti u Furijerov red



Dobijeni rezultat iskoristiti za sumiranje reda

$$\frac{1}{1\cdot 3}-\frac{3}{5\cdot 7}+\dots+\frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)}+\dots$$

Rj. Prikazana f-ja je periodična, perioda 2π pa je možemo pretvoriti u Furijer-ov red. Datu f-ju označimo sa $y=f(x)$. Primjetimo sa slike da imamo $f(\frac{\pi}{3})=\frac{\sqrt{3}}{2}$, $f(\frac{\pi}{2})=\frac{\sqrt{2}}{2}$ $f(\frac{2\pi}{3})=\frac{1}{2}$, $f(\pi)=0$.
Kako je $\cos(\frac{\pi}{6})=\frac{\sqrt{3}}{2}$, $\cos(\frac{\pi}{4})=\frac{\sqrt{2}}{2}$, $\cos(\frac{\pi}{3})=\frac{1}{2}$, $\cos(\frac{\pi}{2})=0$ to možemo primjetiti da je data f-ja $y=\cos(\frac{x}{2})$, $0\le x\le 2\pi$
Trigonometriski red oblika $\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$ $x\in[a,b]$ nazivamo Furijer-ov red na intervalu $[a,b]$ gdje su

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx,\quad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,\quad b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx,$$

$n=1,2,\dots$ Furijer-ovi koeficijenti. U našem slučaju posmatramo interval $[0,2\pi]$ pa imamo $b-a=2\pi$, $\frac{2}{b-a}=\frac{1}{\pi}$, $\frac{2n\pi x}{b-a}=nx$

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\quad a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\quad b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\cos\frac{x}{2}\,dx = \frac{1}{\pi}\,2\int_0^{2\pi}\cos\frac{x}{2}\,d\left(\frac{x}{2}\right) = \frac{2}{\pi}\sin\frac{x}{2}\Big|_0^{2\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \left|\begin{array}{l}\text{data}\\ \text{f-ja } f(x)\\ \text{je periodična}\end{array}\right| = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \left|\begin{array}{l}\text{prema datoj slici}\\ f(x) \text{ je neparna.}\\ \text{Kako je } \cos nx\\ \text{parna to je}\\ f(x)\cos nx \text{ neparna}\\ \text{f-ja, i imamo}\\ \text{simetričan interval}\end{array}\right|$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\cos\left(\frac{x}{2}\right)\sin nx\,dx = \left|\begin{array}{l}\sin(A+B) = \sin(A)\cos B + \sin B\cos A\\ \sin(A-B) = \sin A\cos B - \sin B\cos A\\ \hline 2\sin A\cos B = \sin(A+B) + \sin(A-B)\end{array}\right|$$

$$= \frac{1}{\pi}\cdot\frac{1}{2}\int_0^{2\pi}\left(\sin\left(nx+\frac{x}{2}\right) + \sin\left(nx-\frac{x}{2}\right)\right)dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\sin\left(n+\tfrac{1}{2}\right)x + \sin\left(n-\tfrac{1}{2}\right)x\right)dx$$

$$= \frac{1}{2\pi}\cdot\frac{(-1)}{n+\frac{1}{2}}\cos\left(n+\tfrac{1}{2}\right)x\Big|_0^{2\pi} + \frac{1}{2\pi}\cdot\frac{(-1)}{n-\frac{1}{2}}\cos\left(n-\tfrac{1}{2}\right)x\Big|_0^{2\pi} =$$

$$= \frac{(-1)}{\pi(2n+1)}\left(\overbrace{\cos(2n+1)\pi - \cos 0}^{-2}\right) + \frac{(-1)}{\pi(2n-1)}\left(\overbrace{\cos(2n-1)\pi - \cos 0}^{-2}\right) = \frac{2}{\pi(2n+1)} + \frac{2}{\pi(2n-1)}$$

$$= \frac{8n}{\pi(2n-1)(2n+1)}$$

Prema tome $\cos\frac{x}{2} \sim \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+1)}\sin nx$ traženi Furijerov red

Ako za x uzmemo $\frac{\pi}{2}$ imamo

$$\frac{8}{\pi}\left(\frac{1}{1\cdot 3} + \frac{2\cdot 0}{3\cdot 5} + \frac{3\cdot(-1)}{5\cdot 7} + \dots + \frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)} + \dots\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Prema tome

$$\frac{1}{1\cdot 3} - \frac{3}{5\cdot 7} + \dots + \frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)} + \dots = \frac{\pi\sqrt{2}}{16}$$ tražena suma

Izračunati $I = \iint\limits_D x\,dx\,dy$ gdje je

$$D = \{(x,y) : 1 \le x^2+y^2 \le 4y \ \wedge \ x \le y \ \wedge \ x \ge 0\}.$$

Rj.

$1 \le x^2+y^2$

$x^2+y^2=1$

krug sa centrom $C(0,0)$ poluprečnika 1

$x^2+y^2 \le 4y$

$x^2+y^2=4y$

$x^2+y^2-2\cdot y\cdot 2+4=4$

$x^2+(y-2)^2=4$

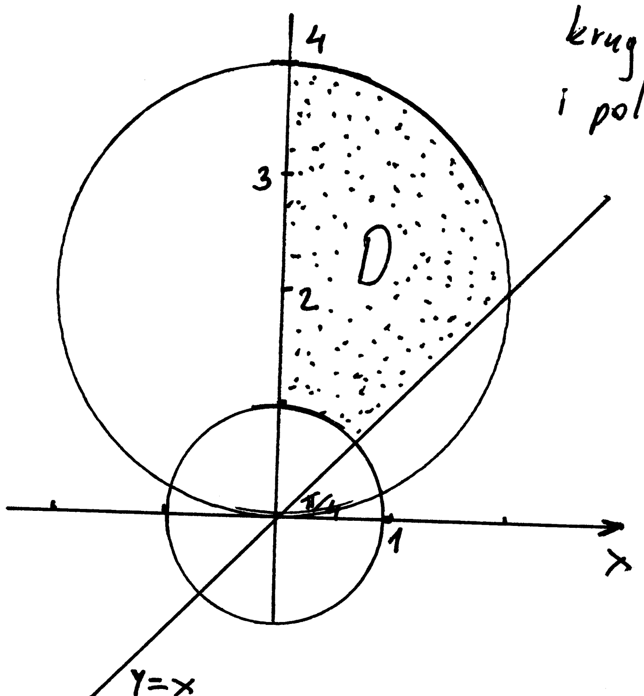
krug sa centrom $C(0;2)$ i poluprečnikom $r=2$

$x \le y$

$x=y$

$x \ge 0$

$x=0$



uvedimo polarne koordinate

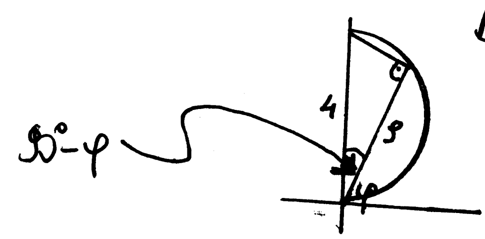
$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$dx\,dy = \rho\,d\rho\,d\varphi$

$D \xrightarrow{\text{transform.}} D'$

$$D' : \begin{cases} \frac{\pi}{4} \le \varphi \le \frac{\pi}{2} \\ 1 \le \rho \le 4\sin\varphi \end{cases}$$



$\underbrace{\cos(90^\circ-\varphi)}_{=\sin\varphi} = \frac{\rho}{4} \Rightarrow \sin\varphi = \frac{\rho}{4}$

$$I = \iint\limits_D x\,dx\,dy = \left|\begin{array}{l}\text{uvodimo}\\ \text{polarne}\\ \text{koordinate}\end{array}\right| = \iint\limits_{D'} \rho\cos\varphi\,\rho\,d\rho\,d\varphi = \int_{\pi/4}^{\pi/2} \cos\varphi \int_1^{4\sin\varphi} \rho^2\,d\rho$$

$$= \frac{1}{3}\int_{\pi/4}^{\pi/2} \cos\varphi\,\underbrace{\rho^3\Big|_1^{4\sin\varphi}}_{64\sin^3\varphi-1} d\varphi = \frac{64}{3}\int_{\pi/4}^{\pi/2} \sin^3\varphi\cos\varphi\,d\varphi - \frac{1}{3}\int_{\pi/4}^{\pi/2} \cos\varphi\,d\varphi = \ldots = \frac{\sqrt{2}}{6} + \frac{11}{3}$$

traženo rješenje

(#) Izračunati krivolinijski integral druge vrste

$$I = \oint_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$$ gdje je C krug

$x^2+y^2+z^2=a^2$ $(a>0)$, $y = x\,\mathrm{tg}\,\alpha$, $(0<\alpha<\frac{\pi}{2})$ uzet u smjeru suprotnom kretanju kazaljke na satu ako se posmatra sa pozitivnog dijela x-ose.

Rj.

$$C: \begin{cases} x^2+y^2+z^2=a^2, (a>0) \\ y = x\,\mathrm{tg}\,\alpha \quad (0<\alpha<\frac{\pi}{2}) \end{cases}$$

Parametrizujmo krivu C. Kako je $\mathrm{tg}\,\alpha = \frac{\sin\alpha}{\cos\alpha}$ to možemo npr. uzeti $x = a\cos\alpha\sin\varphi$. Tada,

$$y = x\,\mathrm{tg}\,\alpha = a\cancel{\cos\alpha}\sin\varphi\frac{\sin\alpha}{\cancel{\cos\alpha}} = a\sin\alpha\sin\varphi$$

Dalje iz $x^2+y^2+z^2=a^2$ imamo

$$(a\cos\alpha\sin\varphi)^2 + (a\sin\alpha\sin\varphi)^2 + z^2 = a^2$$

$$a^2\underbrace{(\cos^2\alpha+\sin^2\alpha)}_{=1}\sin^2\varphi + z^2 = a^2$$

$$z^2 = a^2 - a^2\sin^2\varphi$$

$$z^2 = a^2(1-\sin^2\varphi) \Rightarrow z = a\cos\varphi$$

Dati krug C ima sljedeću parametrizaciju

$$C: \begin{cases} x = a\cos\alpha\sin\varphi \\ y = a\sin\alpha\sin\varphi \\ z = a\cos\varphi \\ 0 \le \varphi \le 2\pi \end{cases} \qquad \begin{aligned} dx &= a\cos\alpha\cos\varphi\,d\varphi \\ dy &= a\sin\alpha\cos\varphi\,d\varphi \\ dz &= -a\sin\varphi \end{aligned}$$

$$\oint_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz =$$

$$= \int_0^{2\pi} \Big[(a\sin\alpha\sin\varphi - a\cos\varphi)\, a\cos\alpha\cos\varphi + (a\cos\varphi - a\cos\alpha\sin\varphi)\, a\sin\alpha\cos\varphi$$
$$+ (a\cos\alpha\sin\varphi - a\sin\alpha\sin\varphi)(-a)\sin\varphi \Big]\, d\varphi$$

$$= \boxed{a^2\sin\alpha\cos\alpha\int_0^{2\pi}\sin\varphi\cos\varphi\,d\varphi} - a^2\cos\alpha\int_0^{2\pi}\cos^2\varphi\,d\varphi + a^2\sin\alpha\int_0^{2\pi}\cos^2\varphi\,d\varphi$$

$$\boxed{-a^2\sin\alpha\cos\alpha\int_0^{2\pi}\sin\varphi\cos\varphi\,d\varphi} - a^2\cos\alpha\int_0^{2\pi}\sin^2\varphi\,d\varphi + a^2\sin\alpha\int_0^{2\pi}\sin^2\varphi\,d\varphi$$

$$= -a^2\cos\alpha\int_0^{2\pi}\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{=1}\,d\varphi + a^2\sin\alpha\int_0^{2\pi}\underbrace{(\sin^2\varphi+\cos^2\varphi)}_{=1}\,d\varphi$$

$$= 2\pi a^2(\sin\alpha-\cos\alpha) = 2a^2(\sin\alpha-\cos\alpha)\pi$$

traženo rješenje

(#) Odrediti površinu koju cilindar $x^2+y^2=ax$ isjeca na lopti $x^2+y^2+z^2=a^2$ iznad ravni Oxy.

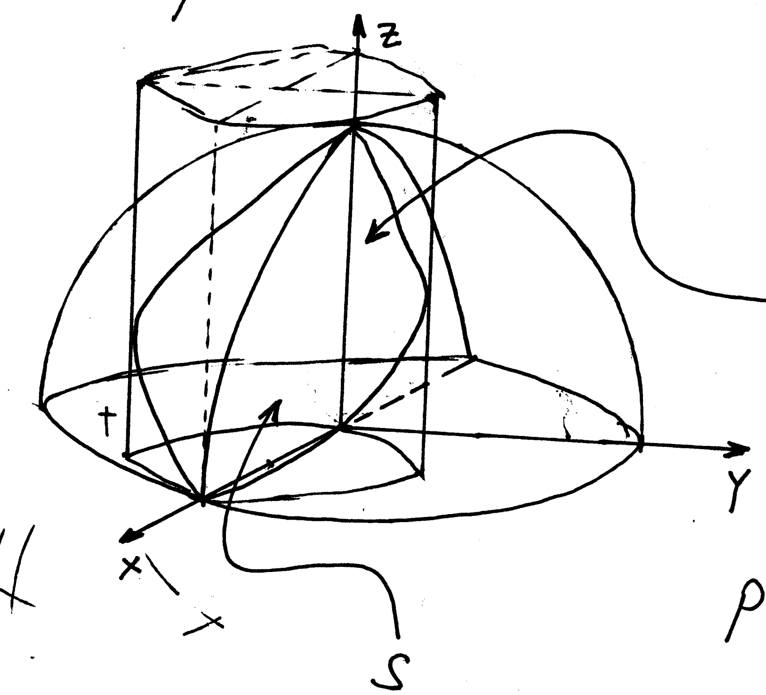
Rj.

Skicirajmo sliku

$$x^2+y^2=ax$$

$$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$



Trebamo izračunati površinu dijela lopte koji se nalazi unutar cilindra.

$$P=\iint_S dS=\iint_D\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$z^2=a^2-x^2-y^2$$

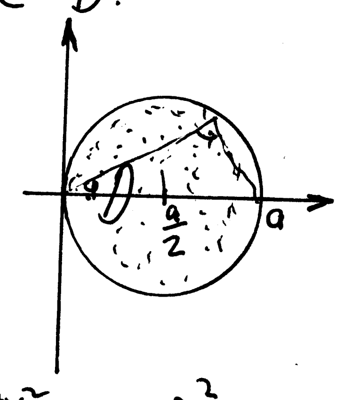
$$z=\pm\sqrt{a^2-x^2-y^2}$$

kako je u pitanju gornji dio polusfere to imamo

$$z=+\sqrt{a^2-x^2-y^2}$$

$$z'_x=\frac{-x}{\sqrt{a^2-x^2-y^2}},\quad z'_y=\frac{-y}{\sqrt{a^2-x^2-y^2}},\quad 1+z'^2_x+z'^2_y=\frac{a^2-x^2-y^2+x^2+y^2}{a^2-x^2-y^2}$$

$$P=\iint_D\sqrt{\frac{a^2}{a^2-x^2-y^2}}\,dx\,dy=a\iint_D\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$$ gdje je D:



Uvedimo polarne koordinate

$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$dx\,dy=\rho\,d\rho\,d\varphi$

$$D\xrightarrow{\text{transformiše}}D':\begin{cases}0\le\rho\le a\cos\varphi\\-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{cases}$$

$\cos\varphi=\frac{\rho}{a}$

$x^2+y^2=\ldots=\rho^2$

$$a\iint\limits_{D}\frac{dx\,dy}{\sqrt{a^2-(x^2+y^2)}}=\left|\begin{array}{l}\text{uvodimo}\\ \text{polarne}\\ \text{koordinate}\end{array}\right|=a\iint\limits_{D'}\frac{\rho\,d\rho\,d\varphi}{\sqrt{a^2-\rho^2}}$$

$$=a\int_{-\pi/2}^{\pi/2}d\varphi\int_{0}^{a\cos\varphi}\frac{\rho\,d\rho}{\sqrt{a^2-\rho^2}}=\left|\begin{array}{l}d(a^2-\rho^2)=-2\rho\,d\rho\\ \rho\,d\rho=-\frac{1}{2}d(a^2-\rho^2)\end{array}\right|=$$

$$=a\int_{-\pi/2}^{\pi/2}d\varphi\int_{0}^{a\cos\varphi}-\frac{1}{2}(a^2-\rho^2)^{-\frac{1}{2}}d(a^2-\rho^2)=-\frac{1}{2}\cdot 2a\int_{-\pi/2}^{\pi/2}\underbrace{(a^2-\rho^2)^{\frac{1}{2}}\Big|_{0}^{a\cos\varphi}}_{\underbrace{(a^2-a^2\cos^2\varphi)^{\frac{1}{2}}}_{(a^2\underbrace{(1-\cos^2\varphi)}_{\sin^2\varphi})^{1/2}}-(a^2-0)^{\frac{1}{2}}}d\varphi$$

$$=-a\int_{-\pi/2}^{\pi/2}(a\sin\varphi-a)\,d\varphi$$

$$=-a^2\int_{-\pi/2}^{\pi/2}(\sin\varphi-1)\,d\varphi=-a^2\cdot\Big(\underbrace{-\cos\varphi\Big|_{-\pi/2}^{\pi/2}}_{0}-\underbrace{\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{\frac{\pi}{2}+\frac{\pi}{2}}\Big)=a^2\pi \quad \text{traženo rješenje}$$